

Each problem below has 10 points. I marked the difficult ones with *.

1*. Exercise 1.6 from RA: Consider a sequence of n flips of an unbiased coin. Let H_i denote the absolute value of the excess of the number of HEADS over the number of TAILS seen in the first i flips. Define $H = \max_i H_i$. Show that $E[H_i] = \Theta(\sqrt{i})$ and that $E[H] = \Theta(\sqrt{n})$. Try to refrain from the use of advanced tools like the Chernoff bound or Martingales. For the second part, you may want to read Section 2.3 of PC for the notion of conditional expectations.

2*. Exercise 1.7(a) from RA: Suppose we choose a permutation π of the ordered set $N = \{1, 2, \dots, n\}$ uniformly at random from the space of all permutations of N . Let $L(\pi)$ denote the length of the longest increasing subsequence in permutation π . For large n and some positive constant c , prove that $E[L(\pi)] \geq c\sqrt{n}$.

3. Exercise 3.9 from PC: (a) Let X be the sum of Bernoulli (i.e., $\{0, 1\}$) random variables, $X = \sum_{i=1}^n X_i$. The X_i do not need to be independent. Show that

$$E[X^2] = \sum_{i=1}^n \Pr(X_i = 1) \cdot E[X|X_i = 1]$$

Hint: Start by showing that

$$E[X^2] = \sum_{i=1}^n E[X_i X]$$

and then apply conditional expectations. (b) Use this equation to provide another derivation for the variance of a binomial random variable with parameters n and p .

4. Exercise 3.19 from PC: Let Y be a nonnegative integer-valued random variable with positive expectation. Prove

$$\frac{E[Y]^2}{E[Y^2]} \leq \Pr[Y \neq 0] \leq E[Y]$$

5. Exercise 3.21 from PC: A fixed point of a permutation $\pi : [1, n] \rightarrow [1, n]$ is a value for which $\pi(x) = x$. Find the variance in the number of fixed points of a permutation chosen uniformly at random from all permutations. (Hint: Let X_i be 1 if $\pi(i) = i$, so $\sum_{i=1}^n X_i$ is the number of fixed points. You cannot use linearity to find $\text{Var}[\sum_{i=1}^n X_i]$, but you can calculate it directly.)

6. Exercise 3.25 from PC: The weak law of large numbers states that, if X_1, X_2, X_3, \dots are independent and identically distributed random variables with mean μ and standard deviation σ , then for any constant $\epsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \Pr \left(\left| \frac{X_1 + X_2 + \dots + X_n}{n} - \mu \right| > \epsilon \right) = 0$$

Use Chebyshev's inequality to prove the weak law of large numbers.