

Each problem below has 10 points. I marked the difficult ones with *.

1. Exercise 4.8 from RA: Consider a BPP algorithm that has an error probability of $1/2 - 1/p(n)$, for some polynomially bounded function $p(n)$ of the input size n . Using the Chernoff bound on the tail of the binomial distribution, show that a polynomial number of independent repetitions of this algorithm suffice to reduce the error probability to $1/2^n$.

2. Exercise 4.14 from PC: Modify the proof of Theorem 4.4 to show the following bound for a weighted sum of Poisson trials. Let X_1, \dots, X_n be independent Poisson trials such that $\Pr(X_i) = p_i$ and let a_1, \dots, a_n be real numbers in $[0, 1]$. Let $X = \sum_{i=1}^n a_i X_i$ and $\mu = E[X]$. Then the following Chernoff bound holds: for any $\delta > 0$,

$$\Pr(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu.$$

3. Exercise 4.11 from RA: The lattice approximation problem is an extension of the set-balancing problem (Example 4.5). As before, we are given an $n \times n$ matrix \mathbf{A} all of whose entries are 0 or 1. In addition, we are given a column vector \mathbf{p} with n entries, all of which are in the interval $[0, 1]$. We wish to find a column vector \mathbf{q} with n entries, all of which are from the set $\{0, 1\}$, so as to minimize $\|\mathbf{A}(\mathbf{p} - \mathbf{q})\|_\infty$. We think of the vector \mathbf{q} as an integer approximation to the given real vector \mathbf{p} , in the sense that $\mathbf{A}\mathbf{q}$ is close to $\mathbf{A}\mathbf{p}$ in every component. This has applications to approximating certain integer programs given solutions to their linear programming relaxations, along the lines of Section 4.3. Derive a bound on $\|\mathbf{A}(\mathbf{p} - \mathbf{q})\|_\infty$ assuming that \mathbf{q} were derived from \mathbf{p} using randomized rounding.

4. Exercise 4.15 from PC: Let X_1, \dots, X_n be independent random variables such that

$$\Pr(X_i = 1 - p_i) = p_i \quad \text{and} \quad \Pr(X_i = -p_i) = 1 - p_i$$

Let $X = \sum_{i=1}^n X_i$. Prove that

$$\Pr(|X| \geq a) \leq 2e^{-2a^2/n}$$

Hint: You may need to assume the inequality

$$p_i e^{\lambda(1-p_i)} + (1 - p_i) e^{-\lambda p_i} \leq e^{\lambda^2/8} \tag{1}$$

This inequality is difficult to prove directly.

5.* In this exercise we generalize Exercise 4 and prove Hoeffding's bound. (a) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if for any x_1, x_2 and $0 \leq \lambda \leq 1$, the following inequality is satisfied:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Show that the function $f(x) = e^{tx}$ is convex for any $t > 0$. (b) Let Z denote a random variable that takes values in $[a, b]$, where $a < 0 < b$, and $E[Z] = 0$. Let X be the random variable with

$$\Pr(X = a) = b/(b - a) \quad \text{and} \quad \Pr(X = b) = -a/(b - a)$$

so that $E[X] = 0$ as well. Show that for any convex function f , we have $E[f(Z)] \leq E[f(X)]$. (c) Finally, let Z_1, \dots, Z_n be independent random variables such that Z_i takes values in $[a_i, b_i]$ and $E[Z_i] = 0$. Following the Chernoff bound proof techniques (as well as inequality (1)) to show that

$$\Pr(X_1 + \dots + X_n \geq t) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

(While this result holds for continuous random variables, you may assume that Z, Z_1, \dots, Z_n are all discrete.)

6.* Let $\epsilon > 0$ be a small positive constant, e.g. 10^{-4} . We construct a regular bipartite graph G as follows. Let U and V be two sets of n vertices each. We pick $\Delta = 48/\epsilon^2$ perfect matching between U and V with replacement, independently and uniformly at random from all possible perfect matchings, and denote their union by G . (Note that if an edge (u, v) appears in multiple perfect matchings then we have parallel edges between u and v in G . Thus, G is a regular bipartite graph with degree Δ .) Use Chernoff bound to show that when n is large enough, with probability $1 - \exp(-\Omega(n))$, the graph G we get satisfies the following property:

$$\text{number of edges between } A \text{ and } B \geq \frac{\Delta|A||B|}{4n}, \quad \text{for all } A \subseteq U \text{ and } B \subseteq V \text{ such that } |A|, |B| \geq \epsilon n.$$