Each problem below has 10 points.

1. Exercise 6.9 of PC: A tournament is a graph on n vertices with exactly one directed edge between each pair of vertices. If vertices represent players, then each edge can be thought of as the result of a match between the two players: the edge points to the winner. A ranking is an ordering of the n players from best to worst (ties are not allowed). Given the outcome of a tournament, one might wish to determine a ranking of the players. A ranking is said to disagree with a directed edge from y to x if y is ahead of x in the ranking (since x beat y in the tournament).

(a) Prove that, for every tournament, there exists a ranking that disagrees with at most 50% of the edges.

(b) Prove that, for sufficiently large n, there exists a tournament such that every ranking disagrees with at least 49% of the edges in the tournament.

2. Exercise 6.10 of PC: A family of subsets $\mathcal{F}$ of $\{1, 2, \ldots, n\}$ is called an antichain if there is no pair of sets A and B in $\mathcal{F}$ satisfying $A \subset B$.

(a) Give an example of $\mathcal{F}$ where $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$.

(b) Let f_k denote the number of sets in $\mathcal{F}$ of size k. Show that

$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1.$$

(Hint: Choose a random permutation of the numbers from 1 to n, and let $X_k = 1$ if the first k numbers in your permutation yield a set in $\mathcal{F}$. If $X = \sum_{k=0}^{n} X_k$, what can you say about X?)

(c) Argue that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for any antichain $\mathcal{F}$.

3. Exercise 6.18 of PC: Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colors, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$, there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different. You may want to let $A_{u,v,c}$ be the event that u and v are both colored with color c and then consider the family of such events.

4. Exercise 5.14 of RA: An (n, m)-safe set instance consists of a universe U of size n, a safe set $S \subseteq U$, and m target sets $T_1, \ldots, T_m \subseteq U$ such that $|S| = |T_1| = \cdots = |T_m|$ and $S \cap T_i = \emptyset$ for all $i : 1 \leq i \leq m$. An isolator for a safe set instance is a set $I \subseteq U$ that intersects all the target sets but not the safe set. An (n, m)-universal isolating family $\mathcal{F}$ is a collection of subsets of U such that $\mathcal{F}$ contains an isolator for any (n, m)-safe set instance. Show that there exists a (n, m)-universal isolating family $\mathcal{F}$ such that $|\mathcal{F}|$ is polynomially bounded in n and m.

5. Exercise 6.5 of RA: Let G be a 3-colorable graph. Consider the following algorithm for coloring the vertices of G with 2 colors so that no triangle of G is monochromatic. The algorithm begins with an arbitrary 2-coloring of G. While there is a monochromatic triangle in G, it arbitrarily chooses one such triangle, and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps

before the algorithm finds a 2-coloring with the desired property.

6. Exercise 6.13 of RA: Consider the two-dimensional mesh: a graph in which each vertex is a point with integer coordinates in the plane, all coordinates being in the interval $[1, n^{1/2}]$. An edge connects two vertices if they differ in one coordinate by 1. Show that the maximum commute time in this graph is $\Theta(n \log n)$.

7. Let G be an (n, d, c)-expander. Show that there exist constants $\beta, \delta > 0$ such that for any "bad" set of vertices B of cardinality at most βn, the following property holds: the probability that, starting from a vertex chosen uniformly at random, a random walk of length ℓ does not visit any vertex outside of B is at most $\exp(-\delta\ell)$.