COMS 6998: Randomness in Computing	Nov 14, 2012

Random Walks

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1 Definitions

G = (E,V) where G is an undirected, connected graph that can have parallel edges.

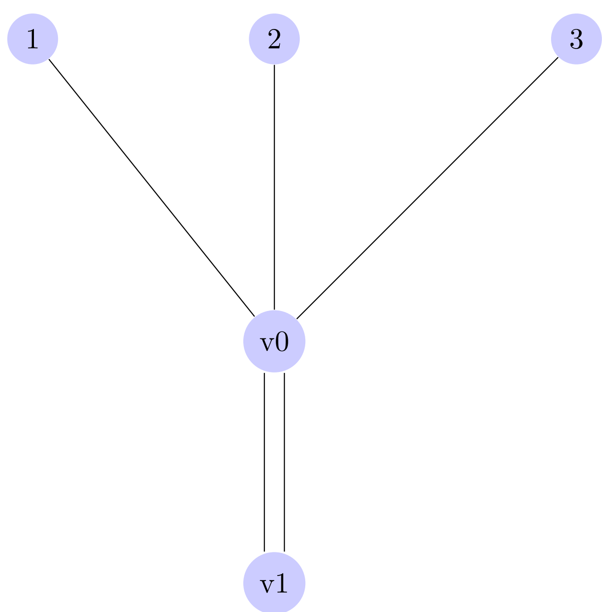
1.1 Basic Definitions

For u,v $\in$ V,

Definition 1. *A random walk on G is the discrete process whereby we start at a vertex v_0, choose a uniformly random neighbor of v_0 called v_1 to take a step to, and then repeat the process from v_1 and so on.*

Observation 2. *The probability of a single step doesn't depend on walk history, only current location.*

Example 3. $Pr[v_0 \to v_1 \mid v_0] = \frac{2}{5}$



Definition 4. *Hitting Time $h_{u,v}$ = the expected number of steps it takes to start from u and reach v the first time on a random walk.*

Observation 5. *$h_{u,v}$ is not symmetric*

Definition 6. *Commute Time $c_{u,v}$ = the expected number of steps it takes to start from u, visit v, and return back to u on a random walk.*

Observation 7. *$c_{u,v}$ is symmetric: $c_{u,v} = h_{u,v} + h_{v,u} = c_{v,u}$*

Definition 8. *Traversal Time c_u = the expected number of steps it takes to start from u and visit all the vertices on a random walk.*

1.2 Extreme Graph Examples

For a graph with n vertices,

Example 9. *If the graph is a path graph P_n where u,v denote the ends of the path: $h_{u,v} = \Theta(n^2)$*

Example 10. *If the graph is a complete graph K_n: $h_{u,v} = \Theta(n)$*

Example 11. *If the graph is a lollipop graph $L_{\frac{n}{2},\frac{n}{2}}$, with u joining the $P_{n/2}$ and the $K_{n/2}$, and v at the other end of the path graph: $h_{u,v} = \Theta(n^3)$ and $h_{v,u} = \Theta(n^2)$*

Claim 12. *$h_{u,v}$ for an $L_{\frac{n}{2},\frac{n}{2}}$ with u and v as described above (Example 11) is the asymptotic worst case for any connected graph.*

2 Brief Applications

2.1 2-SAT Algorithm

Pick an arbitrary assignment for $x = x_1 \ldots x_n$. While x is not a satisfying assignment (if it is we're done), pick one arbitrary unsatisfied clause and randomly set one of its variables to that variable's complement.

Theorem 13. *If $\exists$ a satisfying assignment, then this algorithm finds one in $O(n^2)$ steps in expectation.*

Proof. Let y be a satisfying assignment. Let c(x,y) = the number of variables that x and y agree on. We are done when c(x,y) = n. This problem maps to the path graph example as follows: Let each node 0. . .n on a path graph represent the n+1 possible values for c(x,y) in order, from one end of the graph to the other. When the algorithm flips a variable, the number of correct assignments c(x,y) goes up or down by 1 and we move right or left along the path graph to reflect the change. We move right with probability $\geq 1/2$ since each clause has 2 variables, at least one of which must be incorrect, so we make progress half the time. Assuming (as stated above in Example 9) that $h_{u,v}$ of a path graph is $\Theta(n^2)$, going from c(x,y) = 0 to c(x,y) = n is also $\Theta(n^2)$. ☐

2.2 An s-t Connectivity Algorithm

Given an undirected graph with n vertices, decide if nodes s and t are connected or not.

Theorem 14. *This problem can be solved in randomized log space.*

Proof. Use the following algorithm: Keep track of the random walk index (i.e. the node that you're at) as well as the number of steps taken. If after $\theta(n^3)$ steps you haven't found t yet, output "no connection." (If you come across t before then, then output that there is a connection.) Since we claimed (above, Claim 12) that the worst possible case for $h_{u,v}$ is $\theta(n^3)$, if a path from s to t exists, it is likely to be found within $\theta(n^3)$ steps. ☐

3 Electrical Networks

Given an undirected grapgh G with n nodes and m edges, define N(G), the electrical network of G, as follows: N(G) has a node for every vertex in G, and a 1 ohm resistance between nodes corresponding to the edges in G.

Definition 15. *The resistance between u and v, $R_{u,v}$, where V is the voltage of u with respect to v ($V_u - V_v$) with I amps of current injected into u and removed from v, is*

$$R_{u,v} = R_{eff} = \frac{V}{I}$$

(This follows directly from Ohm's Law: $V = IR$).

Lemma 16. *Let $\phi_{x,v}$ be the voltage at $x \in N(G)$ wrt v, if deg(x) amps are injected into each $x \in V$, and 2m amps are removed from v. Then, $\forall x \in V$,*

$$h_{x,v} = \phi_{x,v}$$

Proof. By the definition of expectation, $\forall x \in V \setminus \{v\}$, hitting time $h_{x,v} = \sum_{w \in \mathcal{N}(x)} \frac{1}{deg(x)}(1 + h_{w,v})$ where $\mathcal{N}(x)$ returns a multiset containing the neighbors of x. Separately, due to Kirchoff's Law (flow into a node = flow out of a node), $\forall x \in V \setminus \{v\}$, $deg(x) = \sum_{w \in \mathcal{N}(x)} (\phi_{x,v} - \phi_{w,v})$. Rearranging the equations shows that they are the same linear system with the same solutions, and one may substitute h with ϕ and vice versa. $\square$

Theorem 17. *For any u,v $\in$ G,*

$$c_{u,v} = 2mR_{u,v}$$

Proof. We know (from Observation 7) that $c_{u,v} = h_{u,v} + h_{v,u}$. We can calculate $-h_{v,u}$ by inverting our circuit N(G). (Remove deg(x) amps from each x, and inject 2m amps into u). By superimposing $N(G)$ and $\bar{N}(G)$, we get a new current I of 2m amps entering u and leaving v, and no net current on any other node in our new graph. The voltage difference ΔV on this graph is thus $h_{u,v} - (-h_{v,u}) = h_{u,v} + h_{v,u} = c_{u,v}$. By Definition 15, $R_{u,v} = \frac{V}{I} = \frac{(h_{u,v} + h_{v,u})}{2m} \Rightarrow c_{u,v} = 2mR_{u,v}$. $\square$

Corollary 18. $c_{u,v} \leq \Theta(n^3)$

Proof. This follows from the above (Theorem 17) as well as the fact that $m \leq \Theta(n^2)$, and $R_{u,v} \leq \Theta(n)$ (since there are at most n nodes on the path from u to v). $\square$

4 Expander Graphs

4.1 Preamble

Let G=(V,E) be a regular, connected graph with parallel edges and degree d.

Definition 19. *For a random walk $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_t \rightarrow \ldots$,*
let the distribution be $q^{(0)} \rightarrow q^{(1)} \rightarrow \ldots \rightarrow q^{(t)} \rightarrow \ldots$.

Definition 20. *Let A be the adjacency matrix of G, where $A_{i,j}$ = the number of edges between v_i and v_j.*

Definition 21. *Let $P = \dfrac{A}{d}$ be the probability matrix of G, and $P_{i,j} = \dfrac{A_{i,j}}{d}$.*

Claim 22. $q^{(t)} = q^{(0)} \times P^t$

Definition 23. *A stationary distribution π is one such that $\pi = \pi p$*

Example 24. *The uniform distribution is stationary: $\pi = (\frac{1}{n} \cdots \frac{1}{n})$*

4.2 Expanders and Eigenvalues

Definition 25. *An (n,d,c)-expander G=(X,Y,E) is a regular, bipartite graph with degree d and $|X| = |Y| = \frac{n}{2}$ such that $\forall S \subset X$,*

$$|N(S)| \geq (1 + c(1 - \frac{2|S|}{n}))|S|$$

$(\geq |S|)$

Example 26. $|S| \leq \frac{n}{3}$
$\Rightarrow |N(S)| \geq (1 + \frac{c}{3})|S|$

Observation 27.

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

Definition 28. *Let A have eigenvalues $d = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = -d$ and corresponding eigenvectors $e_1, e_2, \ldots e_n$ that form an orthonormal basis. From Observation 27, $\lambda_i = -\lambda_{n-i+1}$.*

Definition 29. *Let $\epsilon = \lambda_1 - \lambda_2$*

Theorem 30. *If G is an (n,d,c)-expander,*

$$|\lambda_2| \leq d - \frac{c^2}{1024 + 2c^2}$$

Theorem 31. *If $|\lambda_2| \leq d - \epsilon$,*

$$c \geq \frac{2d\epsilon - \epsilon^2}{d^2}$$

Definition 32. *We will not prove the two above theorems (30 and 31). Instead, we will show a connection between ϵ and Split(G) defined as such:*

$$Split(G) = \min_{S: 1 \leq |S| \leq n-1} \frac{e(S, V \setminus S)}{|S| \cdot |V \setminus S|}$$

Lemma 33. *Equivalent definitions of λ_1 and λ_2 are as follows:*
(note: all norms here are 2-norms)
$\lambda_1 = \max\limits_{x:|x|=1} \{x^T A x\}$
$e_1^T A e_1 = e_1^T \lambda_1 e_1 = \lambda_1 \quad (Ax = \lambda x)$
$\lambda_2 = \max\limits_{\substack{x:|x|=1 \\ \sum x_i = 0}} \{x^T A x\}$

Theorem 34. *Split*$(G) \geq \dfrac{d-\lambda_2}{n}$

Proof. For any S, we show that the ratio in the definition of Split(G) is at least $(d-\lambda_2)/n$. The lemma (33) then follows.
Let $k = |S|$ and define the following vector x:

$$x_i = \begin{cases} \sqrt{\dfrac{n-k}{nk}} & \text{if i} \in \text{S} \\ -\sqrt{\dfrac{k}{n(n-k)}} & \text{if i} \notin \text{S} \end{cases}$$

$\sum x_i = 0$ and $\|x\| = 1$ ✓
$\Rightarrow \lambda_2 \geq x^T Ax = \sum\sum_{(i,j)\in E} x_i x_j$
$= \sum_{(i,j)\in E}(x_i^2 + x_j^2 - (x_i - x_j)^2)$
$= d - \sum_{(i,j)\in E}(x_i - x_j)^2$

$$= d - |e(S, V\setminus S)| \cdot \left(\sqrt{\frac{n-k}{nk}} + \sqrt{\frac{k}{n(n-k)}}\right)^2$$

$$= d - |e(S, V\setminus S)| \cdot \frac{n}{k(n-k)}$$

$$\Rightarrow \lambda_2 \geq d - \frac{e(S, V\setminus S)}{k(n-k)} \cdot n$$

$$\geq \frac{d-\lambda_2}{n}$$

□

Definition 35. *Let A be the adjacency matrix of an (n,d,c)-expander graph G.*
In order to make it such that a random walk on G does not result in a periodic Markov Chain, we reduce all transition probabilities by a factor of 2 and add self loops with probability 1/2 to each vertex.
Therefore, define the transition probability matrix Q, such that it has a stationary distribution, as follows:
$Q = \dfrac{I+P}{2}$ *(pick self-loop with $\frac{1}{2}$ probability.)*
If λ_i are the eigenvalues of A,
then $\lambda'_i = \dfrac{d+\lambda_i}{2d}$ are the eigenvalues of Q.
Thus $1 = \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n = 0$

Claim 36. $q^{(t)} = q^{(0)}Q^t$

Example 37. *We now show that $q^{(t)}$ approaches the uniform distribution quickly as the number of steps t grows.*
$q^{(0)} = \sum_{i=1}^n c_i e_i = c_1 e_1 + \sum_{i=2}^n c_i e_i$
Let $x = c_1 e_1$ and let $y = \sum_{i=2}^n c_i e_i$
$q^{(t)} = (\sum c_i e_i)Q^t$
$= \sum_{i=1}^n c_i \lambda_i^t e_i$
$= x + \sum_{i=1}^n c_i \lambda_i^t e_i$
Now bound $||q^{(t)} - x||_1$
$= ||\sum c_i \lambda_i^t e_i||_1$
$\leq \sqrt{n}\, ||\sum_{i=1}^n c_i \lambda_i^t e_i||_2$
$= \sqrt{n}\sqrt{(\sum c_i^2 \lambda_i^{2t})}$
$\leq \sqrt{n}\sqrt{\sum c_i^2 \lambda_2^{2t}}$

$$= \sqrt{n}\lambda_2^t \sqrt{\sum c_i^2}$$

$$\leq \sqrt{n}\, \lambda_2^t \, ||q^{(0)}||_2$$