

Random Walks

Instructor: *Xi Chen*Scribes: *Michael Rubin*

1 Definitions

$G = (E, V)$ where G is an undirected, connected graph that can have parallel edges.

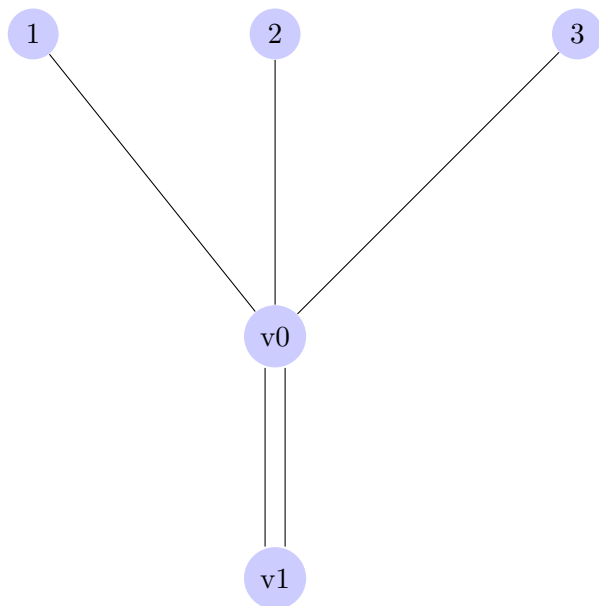
1.1 Basic Definitions

For $u, v \in V$,

Definition 1. A random walk on G is the discrete process whereby we start at a vertex v_0 , choose a uniformly random neighbor of v_0 called v_1 to take a step to, and then repeat the process from v_1 and so on.

Observation 2. The probability of a single step doesn't depend on walk history, only current location.

Example 3. $Pr[v_0 \rightarrow v_1 \mid v_0] = \frac{2}{5}$



Definition 4. Hitting Time $h_{u,v}$ = the expected number of steps it takes to start from u and reach v the first time on a random walk.

Observation 5. $h_{u,v}$ is not symmetric

Definition 6. Commute Time $c_{u,v}$ = the expected number of steps it takes to start from u , visit v , and return back to u on a random walk.

Observation 7. $c_{u,v}$ is symmetric: $c_{u,v} = h_{u,v} + h_{v,u} = c_{v,u}$

Definition 8. *Traversal Time* c_u = the expected number of steps it takes to start from u and visit all the vertices on a random walk.

1.2 Extreme Graph Examples

For a graph with n vertices,

Example 9. If the graph is a path graph P_n where u,v denote the ends of the path: $h_{u,v} = \Theta(n^2)$

Example 10. If the graph is a complete graph K_n : $h_{u,v} = \Theta(n)$

Example 11. If the graph is a lollipop graph $L_{\frac{n}{2}, \frac{n}{2}}$, with u joining the $P_{n/2}$ and the $K_{n/2}$, and v at the other end of the path graph: $h_{u,v} = \Theta(n^3)$ and $h_{v,u} = \Theta(n^2)$

Claim 12. $h_{u,v}$ for an $L_{\frac{n}{2}, \frac{n}{2}}$ with u and v as described above (Example 11) is the asymptotic worst case for any connected graph.

2 Brief Applications

2.1 2-SAT Algorithm

Pick an arbitrary assignment for $x = x_1 \dots x_n$. While x is not a satisfying assignment (if it is we're done), pick one arbitrary unsatisfied clause and randomly set one of its variables to that variable's complement.

Theorem 13. If \exists a satisfying assignment, then this algorithm finds one in $O(n^2)$ steps in expectation.

Proof. Let y be a satisfying assignment. Let $c(x,y)$ = the number of variables that x and y agree on. We are done when $c(x,y) = n$. This problem maps to the path graph example as follows: Let each node $0 \dots n$ on a path graph represent the $n+1$ possible values for $c(x,y)$ in order, from one end of the graph to the other. When the algorithm flips a variable, the number of correct assignments $c(x,y)$ goes up or down by 1 and we move right or left along the path graph to reflect the change. We move right with probability $\geq 1/2$ since each clause has 2 variables, at least one of which must be incorrect, so we make progress half the time. Assuming (as stated above in Example 9) that $h_{u,v}$ of a path graph is $\Theta(n^2)$, going from $c(x,y) = 0$ to $c(x,y) = n$ is also $\Theta(n^2)$. \square

2.2 An s-t Connectivity Algorithm

Given an undirected graph with n vertices, decide if nodes s and t are connected or not.

Theorem 14. This problem can be solved in randomized log space.

Proof. Use the following algorithm: Keep track of the random walk index (i.e. the node that you're at) as well as the number of steps taken. If after $\theta(n^3)$ steps you haven't found t yet, output "no connection." (If you come across t before then, then output that there is a connection.) Since we claimed (above, Claim 12) that the worst possible case for $h_{u,v}$ is $\theta(n^3)$, if a path from s to t exists, it is likely to be found within $\theta(n^3)$ steps. \square

3 Electrical Networks

Given an undirected graph G with n nodes and m edges, define $N(G)$, the electrical network of G , as follows: $N(G)$ has a node for every vertex in G , and a 1 ohm resistance between nodes corresponding to the edges in G .

Definition 15. *The resistance between u and v , $R_{u,v}$, where V is the voltage of u with respect to v ($V_u - V_v$) with I amps of current injected into u and removed from v , is*

$$R_{u,v} = R_{eff} = \frac{V}{I}$$

(This follows directly from Ohm's Law: $V = IR$).

Lemma 16. *Let $\phi_{x,v}$ be the voltage at $x \in N(G)$ wrt v , if $\deg(x)$ amps are injected into each $x \in V$, and $2m$ amps are removed from v . Then, $\forall x \in V$,*

$$h_{x,v} = \phi_{x,v}$$

Proof. By the definition of expectation, $\forall x \in V \setminus \{v\}$, hitting time $h_{x,v} = \sum_{w \in \mathcal{N}(x)} \frac{1}{\deg(x)} (1 + h_{w,v})$ where

$\mathcal{N}(x)$ returns a multiset containing the neighbors of x . Separately, due to Kirchoff's Law (flow into a node = flow out of a node), $\forall x \in V \setminus \{v\}$, $\deg(x) = \sum_{w \in \mathcal{N}(x)} (\phi_{x,v} - \phi_{w,v})$. Rearranging the equations shows

that they are the same linear system with the same solutions, and one may substitute h with ϕ and vice versa. \square

Theorem 17. *For any $u, v \in G$,*

$$c_{u,v} = 2mR_{u,v}$$

Proof. We know (from Observation 7) that $c_{u,v} = h_{u,v} + h_{v,u}$. We can calculate $-h_{v,u}$ by inverting our circuit $N(G)$. (Remove $\deg(x)$ amps from each x , and inject $2m$ amps into u). By superimposing $N(G)$ and $\bar{N}(G)$, we get a new current I of $2m$ amps entering u and leaving v , and no net current on any other node in our new graph. The voltage difference ΔV on this graph is thus $h_{u,v} - (-h_{v,u}) = h_{u,v} + h_{v,u} = c_{u,v}$.

By Definition 15, $R_{u,v} = \frac{V}{I} = \frac{(h_{u,v} + h_{v,u})}{2m} \Rightarrow c_{u,v} = 2mR_{u,v}$. \square

Corollary 18. $c_{u,v} \leq \Theta(n^3)$

Proof. This follows from the above (Theorem 17) as well as the fact that $m \leq \Theta(n^2)$, and $R_{u,v} \leq \Theta(n)$ (since there are at most n nodes on the path from u to v). \square

4 Expander Graphs

4.1 Preamble

Let $G=(V,E)$ be a regular, connected graph with parallel edges and degree d .

Definition 19. *For a random walk $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_t \rightarrow \dots$, let the distribution be $q^{(0)} \rightarrow q^{(1)} \rightarrow \dots \rightarrow q^{(t)} \rightarrow \dots$*

Definition 20. Let A be the adjacency matrix of G , where $A_{i,j}$ = the number of edges between v_i and v_j .

Definition 21. Let $P = \frac{A}{d}$ be the probability matrix of G , and $P_{i,j} = \frac{A_{i,j}}{d}$.

Claim 22. $q^{(t)} = q^{(0)} \times P^t$

Definition 23. A stationary distribution π is one such that $\pi = \pi P$

Example 24. The uniform distribution is stationary: $\pi = (\frac{1}{n} \dots \frac{1}{n})$

4.2 Expanders and Eigenvalues

Definition 25. An (n,d,c) -expander $G=(X,Y,E)$ is a regular, bipartite graph with degree d and $|X| = |Y| = \frac{n}{2}$ such that $\forall S \subset X$,

$$|N(S)| \geq (1 + c(1 - \frac{2|S|}{n}))|S|$$

($\geq |S|$)

Example 26. $|S| \leq \frac{n}{3}$
 $\Rightarrow |N(S)| \geq (1 + \frac{c}{3})|S|$

Observation 27.

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

Definition 28. Let A have eigenvalues $d = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = -d$ and corresponding eigenvectors e_1, e_2, \dots, e_n that form an orthonormal basis. From Observation 27, $\lambda_i = -\lambda_{n-i+1}$.

Definition 29. Let $\epsilon = \lambda_1 - \lambda_2$

Theorem 30. If G is an (n,d,c) -expander,

$$|\lambda_2| \leq d - \frac{c^2}{1024 + 2c^2}$$

Theorem 31. If $|\lambda_2| \leq d - \epsilon$,

$$c \geq \frac{2d\epsilon - \epsilon^2}{d^2}$$

Definition 32. We will not prove the two above theorems (30 and 31). Instead, we will show a connection between ϵ and $Split(G)$ defined as such:

$$Split(G) = \min_{S:1 \leq |S| \leq n-1} \frac{e(S, V \setminus S)}{|S| \cdot |V \setminus S|}$$

Lemma 33. Equivalent definitions of λ_1 and λ_2 are as follows:

(note: all norms here are 2-norms)

$$\lambda_1 = \max_{x:|x|=1} \{x^T Ax\}$$

$$e_1^T A e_1 = e_1^T \lambda_1 e_1 = \lambda_1 \quad (Ax = \lambda x)$$

$$\lambda_2 = \max_{\substack{x:|x|=1 \\ \sum x_i=0}} \{x^T Ax\}$$

Theorem 34. $\text{Split}(G) \geq \frac{d - \lambda_2}{n}$

Proof. For any S , we show that the ratio in the definition of $\text{Split}(G)$ is at least $(d - \lambda_2)/n$. The lemma (33) then follows.

Let $k = |S|$ and define the following vector x :

$$x_i = \begin{cases} \sqrt{\frac{n-k}{nk}} & \text{if } i \in S \\ -\sqrt{\frac{k}{n(n-k)}} & \text{if } i \notin S \end{cases}$$

$$\sum x_i = 0 \text{ and } \|x\| = 1 \quad \checkmark$$

$$\begin{aligned} \Rightarrow \lambda_2 &\geq x^T A x = \sum \sum_{(i,j) \in E} x_i x_j \\ &= \sum_{(i,j) \in E} (x_i^2 + x_j^2 - (x_i - x_j)^2) \\ &= d - \sum_{(i,j) \in E} (x_i - x_j)^2 \\ &= d - |e(S, V \setminus S)| \cdot \left(\sqrt{\frac{n-k}{nk}} + \sqrt{\frac{k}{n(n-k)}} \right)^2 \\ &= d - |e(S, V \setminus S)| \cdot \frac{n}{k(n-k)} \\ \Rightarrow \lambda_2 &\geq d - \frac{e(S, V \setminus S)}{k(n-k)} \cdot n \\ &\geq \frac{d - \lambda_2}{n} \end{aligned}$$

□

Definition 35. Let A be the adjacency matrix of an (n, d, c) -expander graph G .

In order to make it such that a random walk on G does not result in a periodic Markov Chain, we reduce all transition probabilities by a factor of 2 and add self loops with probability 1/2 to each vertex.

Therefore, define the transition probability matrix Q , such that it has a stationary distribution, as follows:

$$Q = \frac{I + P}{2} \quad (\text{pick self-loop with } \frac{1}{2} \text{ probability.})$$

If λ_i are the eigenvalues of A ,

then $\lambda'_i = \frac{d + \lambda_i}{2d}$ are the eigenvalues of Q .

Thus $1 = \lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_n = 0$

Claim 36. $q^{(t)} = q^{(0)} Q^t$

Example 37. We now show that $q^{(t)}$ approaches the uniform distribution quickly as the number of steps t grows.

$$q^{(0)} = \sum_{i=1}^n c_i e_i = c_1 e_1 + \sum_{i=2}^n c_i e_i$$

Let $x = c_1 e_1$ and let $y = \sum_{i=2}^n c_i e_i$

$$q^{(t)} = (\sum c_i e_i) Q^t$$

$$= \sum_{i=1}^n c_i \lambda_i^t e_i$$

$$= x + \sum_{i=1}^n c_i \lambda_i^t e_i$$

Now bound $\|q^{(t)} - x\|_1$

$$= \|\sum c_i \lambda_i^t e_i\|_1$$

$$\leq \sqrt{n} \|\sum_{i=1}^n c_i \lambda_i^t e_i\|_2$$

$$= \sqrt{n} \sqrt{(\sum c_i^2 \lambda_i^{2t})}$$

$$\leq \sqrt{n} \sqrt{\sum c_i^2 \lambda_2^{2t}}$$

$$\begin{aligned} &= \sqrt{n} \lambda_2^t \sqrt{\sum c_i^2} \\ &\leq \sqrt{n} \lambda_2^t \|q^{(0)}\|_2 \end{aligned}$$